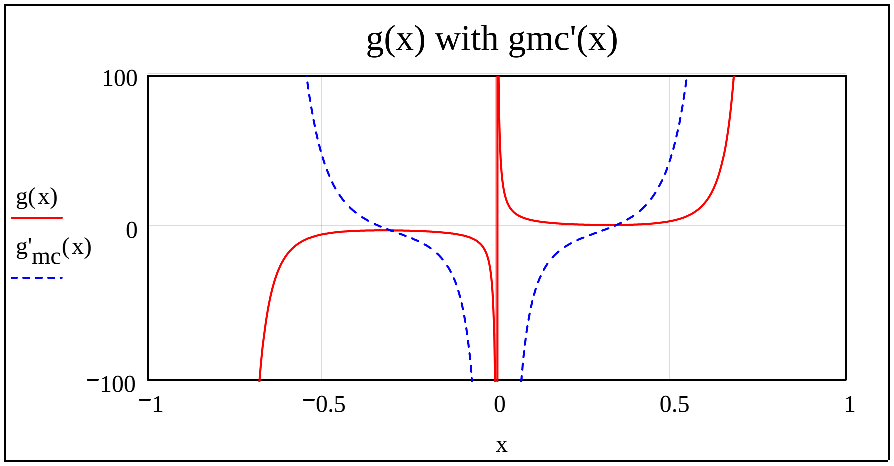
**Problem #1** **Given:** $g(x) := \tan(2x)^3 + \dfrac{1}{\tan(2x)}$

**Mathcad yields:** $g'_{mc}(x) := 3\cdot\tan(2\cdot x)^2\cdot\left(2 + 2\cdot\tan(2\cdot x)^2\right) - \dfrac{1}{\tan(2\cdot x)^2}\cdot\left(2 + 2\cdot\tan(2\cdot x)^2\right)$

**Find:** $\dfrac{d}{dx}g(x)$

graphing domain: $x := -.75, -.7477 .. .75$




differentiating 1st term by hand:

$$\frac{d}{dx}\tan(2x)^3 = 3\cdot\tan(2x)^2\cdot 2\cdot\sec(2x)^2$$

2nd term

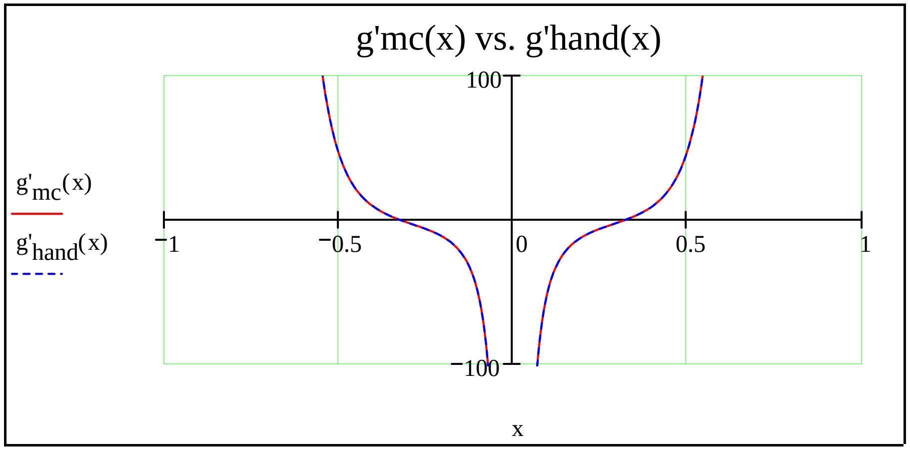
$$\frac{d}{dx}\frac{1}{\tan(2x)} = \frac{-1}{\tan(2x)^2}\cdot 2\cdot\sec(2x)^2$$

yields:

$$g'_{hand}(x) := 3\cdot\tan(2x)^2\cdot 2\cdot\sec(2x)^2 + \frac{-1}{\tan(2x)^2}\cdot 2\cdot\sec(2x)^2$$

The graph of the computer (Mathcad) solution appears the same as the hand solution and graphing the two functions appears to confirm this.

Just for kicks, lets show this is true on the next page.



**Show:** $2\cdot\sec(2\cdot x)^2 = 2 + 2\cdot\tan(2\cdot x)^2$

**Some Identities**

$$\sin(2t) = 2\sin(t)\,\cos(t)$$
$$\cos(2t) = \cos^2(t) - \sin^2(t)$$
$$= 2\cos^2(t) - 1$$
$$= 1 - 2\sin^2(t)$$
$$\tan(2t) = \frac{2\tan(t)}{1-\tan^2(t)}$$

$$1\cdot\left(\frac{1}{\cos(2x)}\right)^2 = 1 + 1\cdot\left(\frac{\sin(2x)}{\cos(2x)}\right)^2$$ divide through by 2

$$\left(\frac{1}{\cos(2x)}\right)^2 = 1 + \left(\frac{2\cdot\sin(x)\cdot\cos(x)}{\cos(2x)}\right)^2$$ identity

$$\left(\frac{1}{\cos(x)^2 - \sin(x)^2}\right)^2 = 1 + \left(\frac{2\cdot\sin(x)\cdot\cos(x)}{\cos(x)^2 - \sin(x)^2}\right)^2$$ identities

$$\left(\frac{1}{\cos(x)^2 - \sin(x)^2}\right)^2 - \left(\frac{2\cdot\sin(x)\cdot\cos(x)}{\cos(x)^2 - \sin(x)^2}\right)^2 = 1$$ collecting terms on left side

$$\frac{1 - (2\cdot\sin(x)\cdot\cos(x))^2}{\left(\cos(x)^2 - \sin(x)^2\right)^2} = 1$$ adding terms (common denominator)

$$\frac{1 - \sin(2x)^2}{\cos(2x)^2} = 1$$ identify

$$1 - \sin(2x)^2 = \cos(2x)^2$$ algebra

$$1 = \sin(2x)^2 + \cos(2x)^2$$ Pythagorean identity

$$1 = 1$$ reflexive property of Real numbers

**Problem #2** **Given:** $f(x) = 2\cdot(3 + \cot(4x))^3$

**Find:** $\frac{d}{dx}f(x)$

since: $\frac{d}{dx}(u\cdot v) = u\cdot\left(\frac{d}{dx}v\right) + v\cdot\left(\frac{d}{dx}u\right)$

letting $u = 2$ $v = (3 + \cot(4x))^3$

$$\frac{d}{dx}f(x) = 2\cdot\frac{d}{dx}(3 + \cot(4x))^3 + (3 + \cot(4x))^3\cdot 0 = 2\cdot\frac{d}{dx}(3 + \cot(4x))^3$$

but ... $\frac{d}{dx}(3 + \cot(4x))^3 = 3\cdot(3 + \cot(4x))^2\cdot\left[\frac{d}{dx}(3 + \cot(4x))\right]$

but ... $\frac{d}{dx}(3 + \cot(4x)) = 0 + \frac{d}{dx}\cot(4x)$

but ... $\frac{d}{dx}\cot(4\cdot x) = -\frac{d}{dx}(4\cdot x)\cdot\csc(4x)^2 = -4\cdot\csc(4x)^2$

so ... $\frac{d}{dx}f(x) = 2\cdot\left[3\cdot(3 + \cot(4x))^2\cdot\left(-4\cdot\csc(4x)^2\right)\right] = 6\cdot(3 + \cot(4x))^2\cdot\left(-4\cdot\csc(4x)^2\right)$

Consider a binomial expansion method

$$(a + b)^3 = a^3 + 3\cdot a^2\cdot b + 3\cdot a\cdot b^2 + b^3$$

letting: $a = 3$ $b = \cot(4x)$

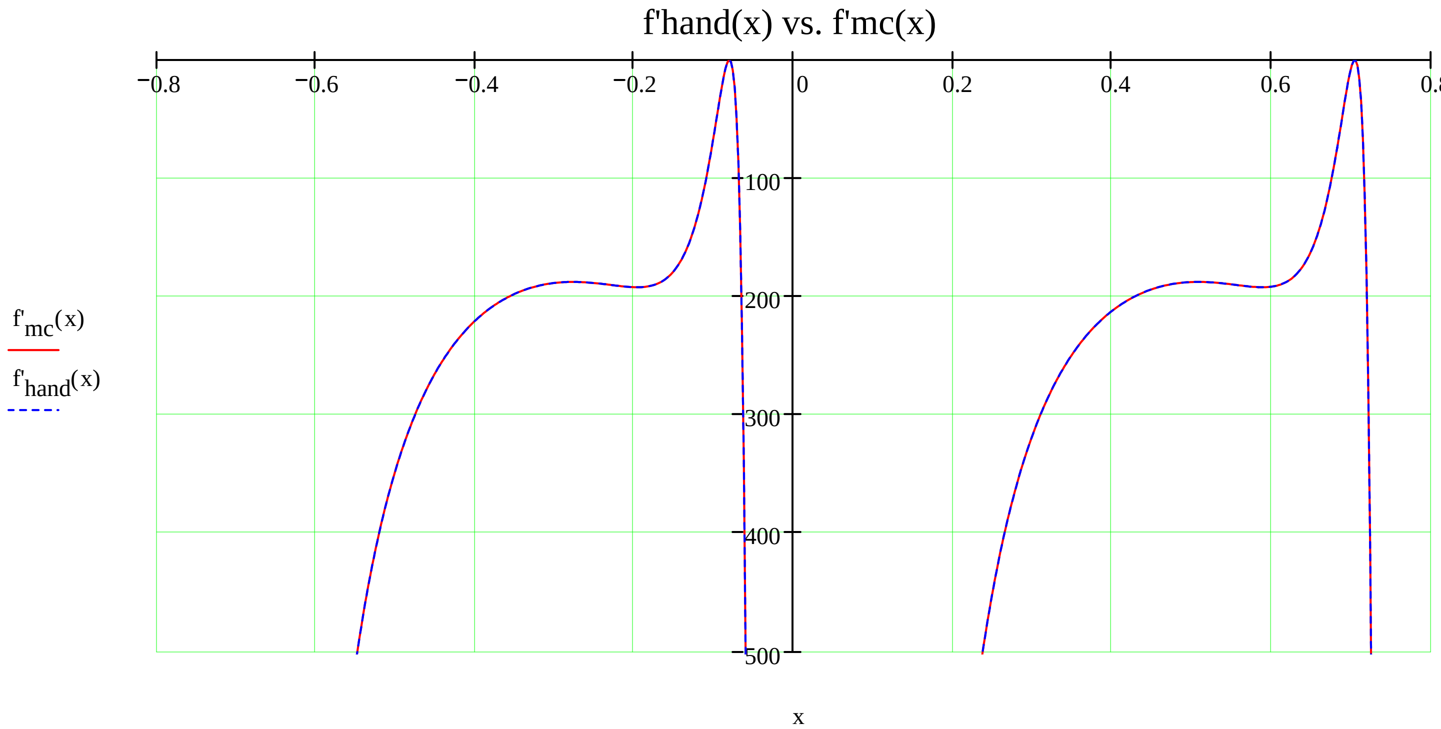
$$2\cdot(3 + \cot(4x))^3 = 2\cdot\left(3^3 + 3\cdot 3^2\cdot\cot(4x) + 3\cdot 3\cdot\cot(4x)^2 + \cot(4x)^3\right)$$

$$2\cdot(3 + \cot(4x))^3 = 54 + 54\cdot\cot(4\cdot x) + 18\cdot\cot(4\cdot x)^2 + 2\cdot\cot(4\cdot x)^3$$

**hand solution:** $f'_{hand}(x) := 6\cdot(3 + \cot(4x))^2\cdot\left(-4\cdot\csc(4x)^2\right)$

**Mathcad solution:** $f'_{mc}(x) := 6\cdot(3 + \cot(4\cdot x))^2\cdot\left(-4 - 4\cdot\cot(4\cdot x)^2\right)$

noting: $\frac{d}{dx}\cot(4x) = -4 - 4\cdot\left(\cot(4\cdot x)^2\right) = -4\cdot\csc(4x)^2$



**Problem #3** **Given:** $h(x) = \cos\left(\frac{3x+1}{x}\right)^3$ Mathcad derivative: $h'_{mc}(x) := -3\cdot\cos\left(\frac{3\cdot x+1}{x}\right)^2\cdot\sin\left(\frac{3\cdot x+1}{x}\right)\cdot\left(\frac{3}{x} - \frac{3\cdot x+1}{x^2}\right)$

**Find:** $\frac{d}{dx}h(x) = h'_{hand'}(x) = 3\cdot\cos\left(\frac{3x+1}{x}\right)^2\cdot\frac{d}{dx}\cos\left(\frac{3x+1}{x}\right)$

but ... $\frac{d}{dx}\cos\left(\frac{3x+1}{x}\right) = -\sin\left(\frac{3x+1}{x}\right)\cdot\left(\frac{d}{dx}\frac{3x+1}{x}\right)$

remembering the quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v}\cdot\left(\frac{d}{dx}u\right) - \frac{u}{v^2}\cdot\left(\frac{d}{dx}v\right)$

but ... $\frac{d}{dx}\frac{3x+1}{x} = \frac{1}{x}\cdot\left[\frac{d}{dx}(3x+1)\right] - \frac{3x+1}{x^2}\cdot\left(\frac{d}{dx}x\right)$

so ... $h'_{hand'}(x) = 3\cdot\cos\left(\frac{3x+1}{x}\right)^2\cdot -\sin\left(\frac{3x+1}{x}\right)\cdot\left[\frac{1}{x}\cdot(3) - \frac{3x+1}{x^2}\cdot(1)\right]$

simplifying $h'_{hand'}(x) = 3\cdot\cos\left(\frac{3x+1}{x}\right)^2\cdot -\sin\left(\frac{3x+1}{x}\right)\cdot\left(\frac{-1}{x^2}\right)$

**Problem #4** **Given:** $q(x) = 2\cdot(3 + \cot(4x))^3$

Mathcad derivative: $q'_{mc}(x) := 6\cdot(3 + \cot(4\cdot x))^2\cdot\left(-4 - 4\cdot\cot(4\cdot x)^2\right)$

**Find:** $\frac{d}{dx}q(x) = q'_{hand}(x) = 2\cdot\left[\frac{d}{dx}(3 + \cot(4x))^3\right] + (3 + \cot(4x))^3\cdot\left(\frac{d}{dx}2\right)$

or $q'_{hand}(x) = 2\cdot\left[\frac{d}{dx}(3 + \cot(4x))^3\right] + 0$

but ... $\left[\frac{d}{dx}(3 + \cot(4x))^3\right] = 3\cdot(3 + \cot(4x))^2\cdot\left[\frac{d}{dx}(3 + \cot(4x))\right]$

but ... $\frac{d}{dx}(3 + \cot(4x)) = 0 + \frac{d}{dx}\cot(4x)$

$\frac{d}{dx}\cot(4x) = -\frac{d}{dx}(4x)\cdot\csc(4x)^2$

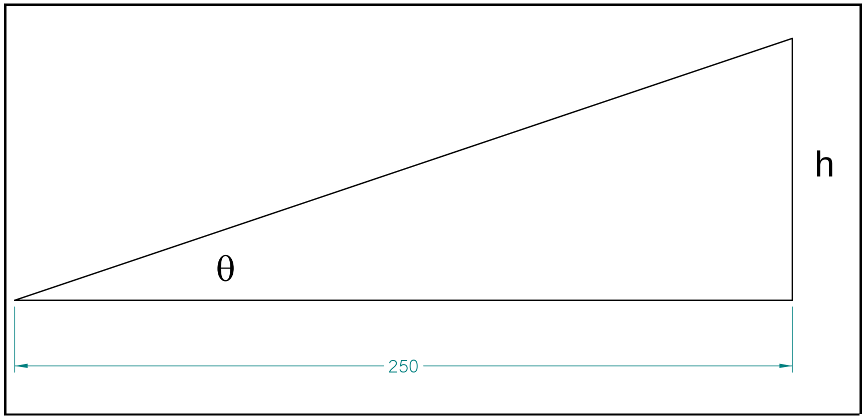
or $\frac{d}{dx}\cot(4x) = -4\cdot\csc(4x)^2$

so ... $q'_{hand}(x) := 6\cdot(3 + \cot(4x))^2\cdot\left(-4\cdot\csc(4x)^2\right)$

(which is an identify to the Mathcad fuction)

**Problem #5** **Given:** A balloon leaves the ground 250' from an observer and rises at a rate of 5 ft/sec.
**Find:** How fast is the angle of elevation increasing after 8 seconds.

We know the height as a function of the rate (r) and time.
Assuming the rate is constant:




$h(r,t) = r \cdot t$ where : $r = 5 \cdot \frac{ft}{sec}$

so ... $h(t) = 5 \cdot \frac{ft}{sec} \cdot t$

and we know that height is a function of $\theta$ and the observer distance (d) to the balloon.
Assuming the distance remains constant:

$\tan(\theta) = \frac{h}{d}$ or $h(d,\theta) = d \cdot \tan(\theta)$ so where $d := 250 \cdot ft$ $h(\theta) = 250 \cdot ft \cdot \tan(\theta)$

Setting the two height equations (each functions of different variables) equal to each other yields: $h(t) = h(\theta)$ or $5 \cdot \frac{ft}{sec} \cdot t = 250 \cdot ft \cdot \tan(\theta)$

Solving for $\theta$ as a function of time (t) yields: $\theta(t) := \operatorname{atan}\left(\frac{1}{50} \cdot \frac{t}{sec}\right)$ So at 8 seconds the angle of elevation is: $\theta(8 \cdot sec) = 9.09\,deg$

Taking the first derivative of $\theta(t)$ with respect to time remembering:

$$\frac{d}{dx}\operatorname{atan}(u) = \frac{1}{1+u^2} \cdot \left(\frac{d}{dx}u\right)$$

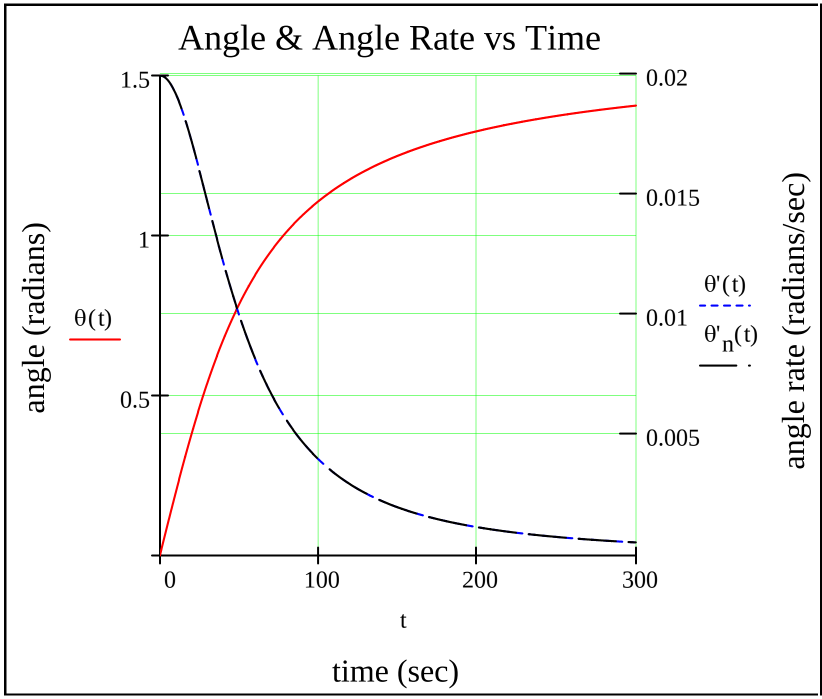
$$\theta'(t) = \frac{d}{dt}\theta(t) = \frac{d}{dt}\operatorname{atan}\left(\frac{t}{50 \cdot sec}\right) = \frac{1}{1+\left(\frac{t}{50 \cdot sec}\right)^2} \cdot \frac{1}{50 \cdot sec} = \frac{50 \cdot sec}{2500 + \frac{t^2}{sec^2}} = 50 \cdot \frac{sec^3}{t^2 + 2500 \cdot sec^2}$$

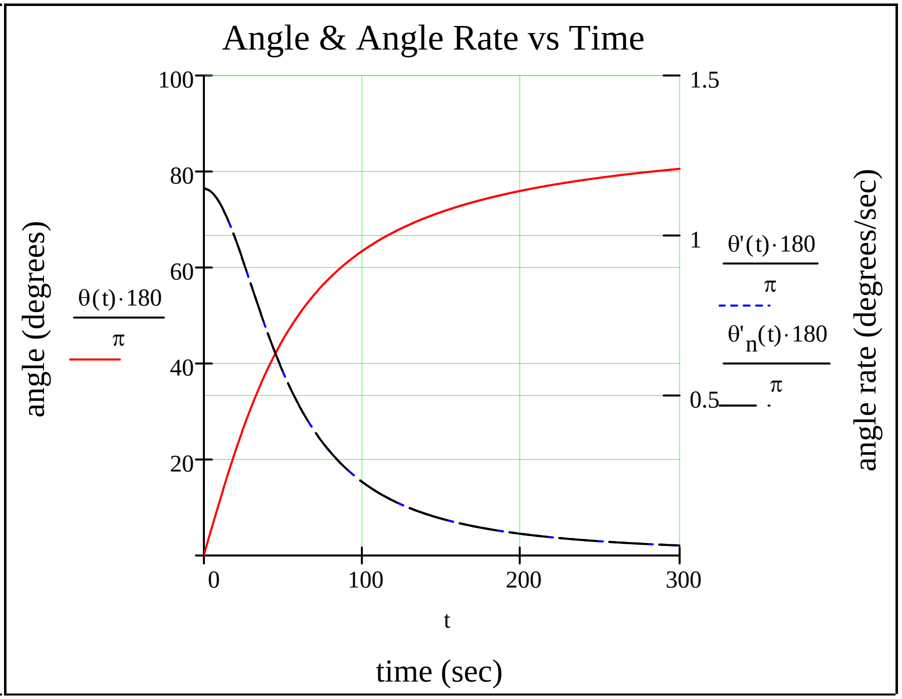
**Graphical investigation:** $\theta(t) := \operatorname{atan}\left(\frac{1}{50}\cdot\frac{t}{sec}\right)$ $\theta'(t) := 50\cdot\frac{sec^3}{t^2 + 2500\cdot sec^2}$

computing the derivative numerically ***(just for fun)*** using Mathcad yields: $\theta'_n(t) := \frac{d}{dt}\theta(t)$

domain:

$t := 0\cdot sec, .1\cdot sec .. 300\cdot sec$




some basic checks using the degree graph around 100 seconds confirm the functions as reasonable.

$$\frac{\theta(101\cdot sec) - \theta(100\cdot sec)}{1\cdot sec} = 0.227363\ \frac{1}{s}\ deg$$

$$\frac{\theta(111\cdot sec) - \theta(110\cdot sec)}{1\cdot sec} = 0.19475\ \frac{1}{s}\ deg$$